FIRST ORDER THEORIES OF INDIVIDUAL CONCEPTS AND PROPOSITIONS

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Abstract

We discuss first order theories in which *individual concepts* are admitted as mathematical objects along with the things that *reify* them. This allows very straightforward formalizations of knowledge, belief, wanting, and necessity in ordinary first order logic without modal operators. Applications are given in philosophy and in artificial intelligence. We do not treat general concepts, and we do not present any full axiomatizations but rather show how various facts can be expressed.

1 Introduction

"...it seems that hardly anybody proposes to use different variables for propositions and for truth-values, or different variables for individuals and individual concepts."—(Carnap 1956, p. 113).

Admitting individual concepts as objects—with concept-valued constants, variables, functions and expressions— allows ordinary first order theories of necessity, knowledge, belief and wanting without modal operators or quotation marks and without the restrictions on substituting equals for equals that either device makes necessary.

In this paper we will show how various individual concepts and propositions can be expressed. We are not yet ready to present a full collection of axioms. Moreover, our purpose is not to explicate what concepts are in a philosophical sense but rather to develop a language of concepts for representing facts about knowledge, belief, etc. in the memory of a computer.

Frege (1892) discussed the need to distinguish direct and indirect use of words. According to one interpretation of Frege's ideas, the meaning of the phrase *"Mike's telephone number"* in the sentence *"Pat knows Mike's telephone number"* is the concept of Mike's telephone number, whereas its meaning in the sentence *"Pat dialed Mike's telephone number"* is the number itself. Thus if we also have *"Mary's telephone number = Mike's telephone number"*, then *"Pat dialed Mary's telephone number"* follows, but *"Pat knows Mary's telephone number* does not.

It was further proposed that a phrase has a *sense* which is a *concept* and is its *meaning* in *oblique contexts* like knowing and wanting, and a *denotation* which is its *meaning*in *direct contexts* like dialing. *Denotations* are the basis of the semantics of first order logic and model theory and are well understood, but *sense* has given more trouble, and the modal treatment of oblique contexts avoids the idea. On the other hand, logicians such as Carnap (1947 and 1956), Church (1951) and Montague (1974) see a need for *concepts* and have proposed formalizations. All these formalizations involve modifying the logic used; ours doesn't modify the logic and is more powerful, because it includes mappings from objects to concepts. Robert Moore's forthcoming dissertation also uses concepts in first order logic.

The problem identified by Frege—of suitably limiting the application of the substitutitivity of equals for equals—arises in artificial intelligence as well as in philosophy and linguistics for any system that must represent information about beliefs, knowledge, desires, or logical necessity—regardless of whether the representation is declarative or procedural (as in PLANNER and other AI formalisms).

Our approach involves treating concepts as one kind of object in an ordinary first order theory. We shall have one term that denotes Mike's telephone number and a different term denoting the concept of Mike's telephone number instead of having a single term whose denotation is the number and whose sense is a concept of it. The relations among concepts and between concepts and other entities are expressed by formulas of first order logic. Ordinary model theory can then be used to study what spaces of concepts satisfy various sets of axioms.

We treat primarily what Carnap calls *individual concepts* like *Mike's telephone number* or *Pegasus* and not general concepts like *telephone* or *unicorn*. Extension to general concepts seems feasible, but individual concepts provide enough food for thought for the present.

This is a preliminary paper in that we don't give a comprehensive set of axioms for concepts. Instead we merely translate some English sentences into our formalism to give an idea of the possibilities.

2 Knowing What and Knowing That

To assert that Pat knows Mike's telephone number we write

$$true\ Know(Pat, Telephone\ Mike) \tag{1}$$

with the following conventions:

1. Parentheses are often omitted for one argument functions and predicates. This purely syntactic convention is not important. Another convention is to capitalize the first letter of a constant, variable or function name when its value is a concept. (We considered also capitalizing the last letter when the arguments are concepts, but it made the formulas ugly).
2. $Mike$ is the concept of Mike; i.e. it is the *sense* of the expression *"Mike"*. $mike$ is Mike himself.
3. $Telephone$ is a function that takes a concept of a person into a concept of his telephone number. We will also use $telephone$ which takes the person himself into the telephone number itself. We do not propose to identify the function $Telephone$ with the general concept of a person's telephone number.
4. If P is a person concept and X is another concept, then $Know(P, X)$ is an assertion concept or *proposition* meaning that P knows the value of X. Thus in (1) $Know(Pat, TelephoneMike)$ is a proposition and not a truth value. Note that we are formalizing *knowing what* rather than *knowing that* or *knowing how*. For AI and for other practical purposes, *knowing what* seems to be the most useful notion of the three. In English, *knowing what* is written *knowing whether* when the "knowand" is a proposition.
5. It is often convenient to write $know(pat, Telephone\ Mike)$ instead of

$$true\ Know(Pat, TelephoneMike)$$

when we don't intend to iterate knowledge further. *know* is a predicate in the logic, so we cannot apply any knowledge operators to it. We will have

$$know(pat, Telephone\ Mike) \equiv true\ Know(Pat, Telephone\ Mike). \quad (2)$$

6. We expect that the proposition $Know(Pat, Telphone\ Mike)$ will be useful accompanied by axioms that allow inferring that Pat will use this knowledge under appropriate circumstances, i.e. he will dial it or retell it when appropriate. There will also be axioms asserting that he will know it after being told it or looking it up in the telephone book.

7. While the sentence "*Pat knows Mike*" is in common use, it is harder to see how $Know(Pat, Mike)$ is to be used and axiomatized. I suspect that new methods will be required to treat knowing a person.

8. *true* Q is the truth value, t or f, of the proposition Q, and we must write *true* Q in order to assert Q. Later we will consider formalisms in which *true* has a another argument—a *situation*, a *story*, a *possible world*, or even a *partial possible world* (a notion we suspect will eventually be found necessary).

9. The formulas are in a sorted first order logic with functions and equality. Knowledge, necessity, etc. will be discussed without extending the logic in any way—solely by the introduction of predicate and function symbols subject to suitable axioms. In the present informal treatment, we will not be explicit about sorts, but we will use different letters for variables of different sorts.

The reader may be nervous about what is meant by *concept*. He will have to remain nervous; no final commitment will be made in this paper. The formalism is compatible with many possibilities, and these can be compared using the models of their first order theories. Actually, this paper isn't much motivated by the philosophical question of what concepts really are. The goal is more to make a formal structure that can be used to represent facts about knowledge and belief so that a computer program can reason about who has what knowledge in order to solve problems. From either the philosophical or the AI point of view, however, if (1) is to be reasonable, it must not follow from (1) and the fact that Mary's telephone number is the same as Mike's, that Pat knows Mary's telephone number.

The proposition that Joe knows *whether* Pat knows Mike's telephone number, is written

$$Know(Joe, Know(Pat, Telephone\ Mike)) \tag{3}$$

and asserting it requires writing

$$true\ Know(Joe, Know(Pat, Telephone\ Mike)) \tag{4}$$

while the proposition that Joe knows *that* Pat knows Mike's telephone number is written

$$K(Joe, Know(Pat, Telephone\ Mike) \tag{5}$$

where $K(P, Q)$ is the proposition that P knows *that* Q. English does not treat knowing a proposition and knowing an individual concept uniformly; knowing an individual concept means knowing its value while knowing a proposition means knowing that it has a particular value, namely t. There is no reason to impose this infirmity on robots.

We first consider systems in which corresponding to each concept X, there is a thing x of which X is a concept. Then there is a function *denot* such that

$$x = denot\ X. \tag{6}$$

Functions like $Telephone$ are then related to *denot* by equations like

$$(\forall P1\ P2)(denot\ P1 = denot\ P2 \supset denot\,Telephone\ P1 = denot\,Telephone\ P2). \tag{7}$$

We call $denot X$ the *denotation* of the concept X, and (7) asserts that the denotation of the concept of P's telephone number depends only on the denotation of the concept P. The variables in (7) range over concepts of persons, and we regard (7) as asserting that $Telephone$ is *extensional* with respect to *denot*. Note that our *denot* operates on concepts rather than on expressions; a theory of expressions will also need a denotation function. From (7) and suitable logical axioms follows the existence of a function *telephone* satisfying

$$(\forall P)(denot\ Telephone\ P = telephone\ denot\ P). \tag{8}$$

Know is extensional with respect to *denot* in its first argument, and this is expressed by

$$(\forall P1\ P2)(denot\ P1 = denot\ P2 \supset denot\ Know(P1, X) = denot\ Know(P2, X)), \quad (9)$$

but it is Not extensional in its second argument. We can therefore define a predicate $know(p, X)$ satisfying

$$(\forall P\ \ X)(true\ \ Know(P, X) \equiv know(denot\ \ P, X)). \qquad (10)$$

(Note that all these predicates and functions are entirely extensional in the underlying logic, and the notion of extensionality presented here is relative to *denot*.)

The predicate *true* and the function *denot* are related by

$$(\forall Q)(true\ \ Q \equiv (denot\ \ Q = t)) \qquad (11)$$

provided truth values are in the range of *denot*, and *denot* could also be provided with a *(partial) possible world* argument.

When we don't assume that all concepts have denotations, we use a predicate $denotes(X, x)$ instead of a function. The extensionality of *Telephone* would then be written

$$\begin{aligned}(\forall P1\ \ P2\ \ x\ \ u)\quad &(denotes(P1, x) \wedge denotes(P2, x) \wedge denotes(Telephone\ \ P1, u)\\ &\supset denotes(Telephone\ \ P2, u))\end{aligned} \qquad (12)$$

We now introduce the function *Exists* satisfying

$$(\forall X)(true\ \ Exists\ \ X \equiv (\exists x) denotes(X, x)) \qquad (13)$$

Suppose we want to assert that Pegasus is a horse without asserting that Pegasus exists. We can do this by introducing the predicate *Ishorse* and writing

$$true\ \ Ishorse\ \ Pegasus \qquad (14)$$

which is related to the predicate *ishorse* by

$$(\forall X\ \ x)(denotes(X, x) \supset (ishorse\ \ x \equiv true\ \ Ishorse\ \ X)) \qquad (15)$$

In this way, we assert extensionality without assuming that all concepts have denotations. *Exists* is extensional in this sense, but the corresponding predicate *exists* is identically true and therefore dispensable.

In order to combine concepts propositionally, we need analogs of the propositional operators such as $\wedge$, etc. which we will write *And*, etc., write as infixes, and axiomatize by

$$(true(Q1\ And\ Q2) \equiv true\ Q1 \wedge true\ Q2), \tag{16}$$

etc. The corresponding formulas for Or, Not, $Implies$, and $Equiv$ are

$$(\forall Q1\ Q2)(true(Q1\ Or\ Q2) \equiv true\ Q1 \vee true\ Q2), \tag{17}$$

$$(true(Not\ Q) \equiv \neg true\ Q), \tag{18}$$

$$(true(Q1\ Implies\ Q2) \equiv trueQ1 \supset true\ Q2) \tag{19}$$

and

$$(true(Q1\ Equiv\ Q2) \equiv (true\ Q1 \equiv true\ Q2)). \tag{20}$$

The equality symbol "=" is part of the logic so that $X = Y$ asserts that X and Y are the same concept. To write propositions expressing equality of the denotations of concepts, we introduce $Equal(X, Y)$ which is the proposition that X and Y denote the same thing if anything. We shall want axioms [1]

$$(\forall X)(true\ Equal(X, X)), \tag{21}$$

$$(\forall X\ Y)(true\ Equal(X, Y) \equiv true\ Equal(Y, X)) \tag{22}$$

and

$$(\forall X\ Y\ Z)(true\ Equal(X, Y) \wedge true\ Equal(Y, Z) \supset true\ Equal(X, Z)), \tag{23}$$

making $true\ Equal(X, Y)$ an equivalence relation, and

$$(\forall X\ Y\ x)(true\ Equal(X, Y) \wedge denotes(X, x) \supset denotes(Y, x)), \tag{24}$$

which relates it to equality in the logic.

We can make the concept of equality *essentially* symmetric by replacing (22) by

$$(\forall X\ Y)(Equal(X, Y) = Equal(Y, X)), \tag{25}$$

i.e. making the two expressions denote the *same concept*.

[1] 1995: I should have used an infixed *Equal* here.

The statement that Mary has the same telephone as Mike is asserted by

$$true\ Equal(Telephone\ Mary, Telephone\ Mike) \tag{26}$$

and it obviously doesn't follow from this and (1) that

$$true\ Know(Pat, Telephone\ Mary) \tag{27}$$

To draw this conclusion we need something like

$$true\ K(Pat, Equal(Telephone\ Mary, Telephone\ Mike)) \tag{28}$$

and suitable axioms about knowledge.

If we were to adopt the convention that a proposition appearing at the outer level of a sentence is asserted and were to regard the denotation-valued function as standing for the sense-valued function when it appears as the second argument of $Know$, we would have a notation that resembles ordinary language in handling obliquity entirely by context. There is no guarantee that general statements could be expressed unambiguously without circumlocution; the fact that the principles of intensional reasoning haven't yet been stated is evidence against the suitability of ordinary language for stating them.

3 Functions from Things to Concepts of them

While the relation $denotes(X, x)$ between concepts and things is many-one, functions going from things to certain concepts of them seem useful. Some things such as numbers can be regarded as having *standard* concepts. Suppose that $Concept1\ n$ gives a standard concept of the number n, so that

$$(\forall n)(denot\ Concept1\ n = n) \tag{29}$$

We can then have simultaneously

$$true\ Not\ Knew(Kepler, Number\ Planets) \tag{30}$$

and

$$true\ Knew(Kepler, Composite\ Concept1\ denot\ Number\ Planets). \tag{31}$$

(We have bravely used $Knew$ instead of $Know$, because we are not now concerned with formalizing tense.) (31) can be condensed using $Composite1$ which takes a number into the proposition that it is composite, i.e.

$$(\forall n)(Composite1\ n = Composite\ Concept1\ n), \tag{32}$$

getting

$$true\ Knew(Kepler, Composite1\ denot\ Number\ Planets). \tag{33}$$

A further condensation can be achieved using $Composite2$ defined by

$$(\forall N)(Composite2\ N = Composite\ Concept1\ denot\ N), \tag{34}$$

letting us write

$$true\ Knew(Kepler, Composite2\ Number\ Planets), \tag{35}$$

which is true even though

$$true\ Knew(Kepler, Composite\ Number\ Planets) \tag{36}$$

is false. (36) is our formal expression of *"Kepler knew that the number of planets is composite"*, while (31), (33), and (35) each expresses a proposition that can only be stated awkwardly in English, e.g. as *"Kepler knew that a certain number is composite, where this number (perhaps unbeknownst to Kepler) is the number of planets"*.

We may also want a map from things to concepts of them in order to formalize a sentence like, *"Lassie knows the location of all her puppies"*. We write this

$$(\forall x)(ispuppy(x, lassie) \supset true\ Knowd(Lassie, Locationd\ Conceptd\ x)). \tag{37}$$

Here $Conceptd$ takes a puppy into a dog's concept of it, and $Locationd$ takes a dog's concept of a puppy into a dog's concept of its location. The axioms satisfied by $Knowd$, $Locationd$ and $Conceptd$ can be tailored to our ideas of what dogs know.

A suitable collection of functions from things to concepts might permit a language that omitted some individual concepts like $Mike$ (replacing it with $Conceptx\ mike$) and wrote many sentences with quantifiers over things rather than over concepts. However, it is still premature to apply Occam's razor. It may be possible to avoid concepts as objects in expressing particular facts but impossible to avoid them in stating general principles.

4 Relations between Knowing What and Knowing That

As mentioned before, *"Pat knows Mike's telephone number"* is written

$$true\ Know(Pat, Telephone\ Mike). \tag{38}$$

We can write *"Pat knows Mike's telephone number is 333-3333"*

$$trueK(Pat, Equal(Telephone\ Mike, Concept1\ \text{“}333{-}3333''))\text{,}(39)$$

where $K(P, Q)$ is the proposition that $denot(P)$ knows the proposition Q and $Concept1(\text{“}333 - 3333'')$ is some standard concept of that telephone number.

The two ways of expressing knowledge are somewhat interdefinable, since we can write

$$(\forall P\ Q)(K(P, Q) = (Q\ And\ Know(P, Q))) \tag{40}$$

and

$$(\forall P\ X)(true\ Know(P, X) \equiv (\exists A)(constant\ A \wedge true\ K(P, Equal(X, A)))).(41)$$

Here *constant* A asserts that A is a constant, i.e. a concept such that we are willing to say that P knows X if he knows it equals A. This is clear enough for some domains like integers, but it is not obvious how to treat knowing a person.

Using the *standard concept* function $Concept1$, we might replace (41) by

$$(\forall P\ X(true\ Know(P, X) \equiv (\exists a)(true\ K(P, Equal(X, Concept1\ a))))(42)$$

with similar meaning.[2]

(41) and (42) express a *denotational* definition of $Know$ in terms of K. A *conceptual* definition seems to require something like

$$(\forall P\ X)(Know(P, X) = Exists\ X\ And\ K(P, Equal(X, Concept2\ denot\ X))),(43)$$

where $Concept2$ is a suitable function from things to concepts and may not be available for all sorts of objects.[3]

[2] 1995: This idea is used in my Elephant 2000 paper to discuss the notion of a responsive answer to a question.

[3] 1995: At present I don't see why $Concept2$ needs to be different from $Concept1$.

5 Replacing Modal Operators by Modal Functions

Using concepts we can translate the content of modal logic into ordinary logic. We need only replace the *modal operators* by *modal functions.* The axioms of modal logic then translate into ordinary first order axioms. In this section we will treat only *unquantified modal logic*. The arguments of the modal functions will not involve quantification although quantification occurs in the outer logic.

$Nec\ Q$ is the proposition that the proposition Q is necessary, and $Poss\ Q$ is the proposition that it is possible. To assert necessity or possibility we must write $true\ Nec\ Q$ or $true\ Poss\ Q$. This can be abbreviated by defining $nec\ Q \equiv true\ Nec\ Q$ and $poss\ Q$ correspondingly. However, since nec is a predicate in the logic with t and f as values, $nec\ Q$ cannot be an argument of nec or Nec.

Before we even get to modal logic proper we have a decision to make—shall $Not\ Not\ Q$ be considered the same proposition as Q, or is it merely extensionally equivalent? The first is written

$$(\forall Q)(Not\ Not\ Q = Q) \tag{44}$$

and the second

$$(\forall Q)(true\ Not\ Not\ Q \equiv true\ Q). \tag{45}$$

The second follows from the first by substitution of equals for equals, but the converse needn't hold.

In *Meaning and Necessity*, Carnap takes what amounts to the first alternative, regarding concepts as L-equivalence classes of expressions. This works nicely for discussing necessity, but when he wants to discuss knowledge without assuming that everyone knows Fermat's last theorem if it is true, he introduces the notion of *intensional isomorphism* and has knowledge operate on the equivalence classes of this relation.

If we choose the first alternative, then we may go on to identify any two propositions that can be transformed into each other by Boolean identities. This can be assured by a small collection of propositional identities like (44) including associative and distributive laws for conjunction and disjunction, De Morgan's law, and the laws governing the propositions T and F. In the second alternative we will want the extensional forms of the same laws. When we get to quantification a similar choice will arise, but if we choose the first alternative, it will be undecideable whether two expressions denote the same concept. I doubt that considerations of linguistic usage or usefulness in AI will unequivocally recommend one alternative, so both will have to be studied.

Actually there are more than two alternatives. Let M be the free algebra built up from the "atomic" concepts by the concept forming function symbols. If $\equiv\equiv$ is an equivalence relation on M such that

$$(\forall X1\ X2)((X1 \equiv\equiv X2) \supset (true\ X1 \equiv true\ X2)), \tag{46}$$

then the set of equivalence classes under $\equiv\equiv$ may be taken as the set of concepts.

Similar possibilities arise in modal logic. We can choose between the *conceptual identity*

$$(\forall W)(Poss\ Q = Not\ Nec\ Not\ Q) \tag{47}$$

and the weaker extensional axiom

$$(\forall Q)(true\ Poss\ Q \equiv true\ Not\ Nec\ Not\ Q). \tag{48}$$

We will write the rest of our modal axioms in extensional form.

We have

$$(\forall Q)(true\ Nec\ Q \supset true\ Q) \tag{49}$$

and

$$(\forall Q1\ Q2)(true\ Nec\ Q1 \wedge true\ Nec(Q1\ Implies\ Q2) \supset true\ Nec\ Q2) \tag{50}$$

yielding a system equivalent to von Wright's T.[4]

S4 is given by adding

$$(\forall Q)(true\ Nec\ Q \equiv true\ Nec\ Nec\ Q) \tag{51}$$

and S5 by adding

$$(\forall Q)(true\ Poss\ Q \equiv true\ Nec\ Poss\ Q). \tag{52}$$

[4]It seems that something to replace necessitation is needed to get T and likewise for S4 and S5.

Actually, there may be no need to commit ourselves to a particular modal system. We can simultaneously have the functions $NecT$, $Nec4$ and $Nec5$, related by axioms such as

$$(\forall Q)(true\ Nec4\ Q \supset true\ Nec5\ Q), \tag{53}$$

which would seem plausible if we regard S4 as corresponding to provability in some system and S5 as truth in the intended model of the system.

Presumably we shall want to relate necessity and equality by the axiom

$$(\forall X)(true\ Nec\ Equal(X, X)). \tag{54}$$

Certain of Carnap's proposals translate to the stronger relation

$$(\forall X\ Y)(X = Y \equiv true\ Nec\ Equal(X, Y)), \tag{55}$$

which asserts that two concepts are the same if and only if the equality of what they may denote is necessary.

6 More Philosophical Examples—Mostly Well Known

Some sentences that recur as examples in the philosophical literature will be expressed in our notation so the treatments can be compared.

First we have *"The number of planets = 9"* and *"Necessarily 9 = 9"* from which one doesn't want to deduce *"Necessarily the number of planets = 9"*. This example is discussed by Quine (1961) and (Kaplan 1969). Consider the sentences

$$\neg nec\ Equal(Number\ Planets, Concept1\ 9) \tag{56}$$

and

$$nec\ Equal(Concept1\ number\ planets, Concept1\ 9) \tag{57}$$

Both are true. (56) asserts that it is not necessary that the number of planets be 9, and (57) asserts that the number of planets, once determined, is a number that is necessarily equal to 9. It is a major virtue of our formalism that both meanings can be expressed and are readily distinguished. Substitutivity of equals holds in the logic but causes no trouble, because *"The number of planets = 9"* may be written

$$number(planets) = 9, \tag{58}$$

or, using concepts, as

$$true\ Equal(Number\ Planets, Concept1\ 9), \tag{59}$$

and *"Necessarily 9=9"* is written

$$nec\ Equal(Concept1\ 9, Concept1\ 9), \tag{60}$$

and these don't yield the unwanted conclusion.

Ryle used the sentences *"Baldwin is a statesman"* and *"Pickwick is a fiction"* to illustrate that parallel sentence construction does not always give parallel sense. The first can be rendered in four ways, namely $true\ Statesman\ Baldwin$ or $statesman\ denot\ Baldwin$ or $statesman\ baldwin$ or $statesman1\ Baldwin$ where the last asserts that the concept of Baldwin is one of a statesman. The second can be rendered only as as $true\ Fiction\ Pickwick$ or $fiction1\ Pickwick$.

Quine (1961) considers illegitimate the sentence

$$(\exists x)(\text{Philip is unaware that } x \text{ denounced Catiline}) \tag{61}$$

obtained from *"Philip is unaware that Tully denounced Catiline"* by existential generalization. In the example, we are also supposing the truth of *"Philip is aware that Cicero denounced Catiline"*. These sentences are related to (perhaps even explicated by) several sentences in our system. $Tully$ and $Cicero$ are taken as distinct concepts. The person is called $tully$ or $cicero$ in our language, and we have

$$tully = cicero, \tag{62}$$

$$denot\ Tully = cicero \tag{63}$$

and

$$denot\ Cicero = cicero. \tag{64}$$

We can discuss Philip's concept of the person Tully by introducing a function $Concept2(p1, p2)$ giving for some persons $p1$ and $p2$, $p1$'s concept of $p2$. Such a function need not be unique or always defined, but in the present case, some of our information may be conveniently expressed by

$$Concept2(philip, tully) = Cicero, \tag{65}$$

asserting that Philip's concept of the person Tully is *Cicero*. The basic assumptions of Quine's example also include

$$true\ K(Philip, Denounced(Cicero, Catiline)) \tag{66}$$

and

$$\neg true\ K(Philip, Denounced(Tully, Catiline)). \tag{67}$$

[5] From (63), ..., (67) we can deduce

$$(\exists P)(true\ Denounced(P, Catiline)\ And\ Not\ K(Philip, Denounced(P, Catiline))) \tag{68}$$

from (67) and others, and

$$\neg(\exists p)(denounced(p, catiline)$$

$$\wedge$$

$$\neg true\ K(Philip, Denounced(Concept2(philip, p), Catiline))), \tag{69}$$

using the additional hypotheses

$$(\forall p)(denounced(p, catiline) \supset p = cicero), \tag{70}$$

$$denot\ Catiline = catiline \tag{71}$$

and

$$(\forall P1\ P2)(denot\ Denounced(P1, P2) \equiv denounced(denot\ P1, denot\ P2)). \tag{72}$$

Presumably (68) is always true, because we can always construct a concept whose denotation is Cicero unbeknownst to Philip. The truth of (69) depends on Philip's knowing that someone denounced Catiline and on the map $Concept2(p1, p2)$ that gives one person's concept of another. If we refrain from using a silly map that gives something like

[5] 1995: Quine would also want $true\ Not\ K(Philip, Denounced(Tully, Catiline))$.

$Denouncer(Catiline)$ as its value, we can get results that correspond to intuition.

The following sentence attributed to Russell is is discussed by Kaplan: *"I thought that your yacht was longer than it is"*. We can write it

$$true\ Believed(I, Greater(Length\ Youryacht,$$
$$Concept1\ denot\ Length\ Youryacht)), \tag{73}$$

where we are not analyzing the pronouns or the tense, but are using *denot* to get the actual length of the yacht and $Concept1$ to get back a concept of this true length so as to end up with a proposition that the length of the yacht is greater than that number. This looks problematical, but if it is consistent, it is probably useful.

In order to express *"Your yacht is longer than Peter thinks it is."*, we need the expression $Denot(Peter, X)$ giving a concept of what Peter thinks the value of X is. We now write

$$longer(youryacht, denot\ Denot(Peter, Length\ Youryacht)), \tag{74}$$

but I am not certain this is a correct translation.

Quine (1956) discusses an example in which Ralph sees Bernard J. Ortcutt skulking about and concludes that he is a spy, and also sees him on the beach, but doesn't recognize him as the same person. The facts can be expresed in our formalism by equations

$$trueBelieve(Ralph, Isspy\ P1) \tag{75}$$

and

$$true\ Believe(Ralph, Not\ Issp\ P2) \tag{76}$$

where $P1$ and $P2$ are concepts satisfying $denot P1 = ortcutt$ and $denot P2 = ortcutt$. $P1$ and $P2$ are further described by sentences relating them to the circumstances under which Ralph formed them.

We can still consider a simple sentence involving the persons as things—write it $believespy(ralph, ortcutt)$, where we define

$$(\forall p1\ p2)(believespy(p1, p2) \equiv true\ Believe(Concept1\ p1, Isspy\ Concept7\ p2)) \tag{77}$$

using suitable mappings $Concept1$ and $Concept7$ from persons to concepts of persons. We might also choose to define *believespy* in such a way that it requires $true\ Believe(Concept1\ p1, Isspy\ P)$ for several concepts P of $p2$, e.g. the concepts arising from all $p1$'s encounters with $p2$ or his name. In this case

$$believespy(ralph, ortcutt)$$

will be false and so would a corresponding

$$notbelievespy(ralph, ortcutt)$$

. However, the simple-minded predicate *believespy*, suitably defined, may be quite useful for expressing the facts necessary to predict someone's behavior in simpler circumstances.

Regarded as an attempt to explicate the sentence *"Ralph believes Ortcutt is a spy"*, the above may be considered rather tenuous. However, we are proposing it as a notation for expressing Ralph's beliefs about Ortcutt so that correct conclusions may be drawn about Ralph's future actions. For this it seems to be adequate.

7 Propositions Expressing Quantification

As the examples of the previous sections have shown, admitting concepts as objects and introducing standard concept functions makes "quantifying in" rather easy. However, forming propositions and individual concepts by quantification requires new ideas and additional formalism. We are not very confident of the approach presented here.

We want to continue describing concepts within first order logic with no logical extensions. Therefore, in order to form new concepts by quantification and description, we introduce functions *All*, *Exist*, and *The* such that $All(V, P)$ is (approximately) the proposition that *"for all values of V, P is true"*, $Exist(V, P)$ is the corresponding existential proposition, and $The(V, P)$ is the concept of *"the V such that P"*.

Since *All* is to be a function, V and P must be objects in the logic. However, V is semantically a variable in the formation of $All(V, P)$, etc., and we will call such objects *inner variables* so as to distinguish them from variables in the logic. We will use V, sometimes with subscripts, for a logical variable ranging over inner variables. We also need some constant symbols for inner variables (got that?), and we will use doubled letters, sometimes with subscripts, for these. XX will be used for individual concepts, PP for persons, and QQ for propositions.

The second argument of *All* and friends is a "proposition with variables in it", but remember that these variables are inner variables which are constants in the logic. Got that? We won't introduce a special term for them, but will generally allow concepts to include inner variables. Thus concepts now include inner variables like XX and PP, and concept forming functions like *Telephone* and *Know* take as arguments concepts containing internal variables in addition to the usual concepts.

Thus

$$Child(Mike, PP)\ Implies\ Equal(Telephone\ PP, Telephone\ Mike) \tag{78}$$

is a proposition with the inner variable PP in it to the effect that if PP is a child of Mike, then his telephone number is the same as Mike's, and

$$All(PP, Child(Mike, PP)\ Implies\ Equal(Telephone\ PP, Telephone\ Mike)) \tag{79}$$

is the proposition that all Mike's children have the same telephone number as Mike. Existential propositions are formed similarly to universal ones, but the function *Exist* introduced here should not be confused with the function *Exists* applied to individual concepts introduced earlier.

In forming individual concepts by the description function *The*, it doesn't matter whether the object described exists. Thus

$$The(PP, Child(Mike, PP)) \tag{80}$$

is the concept of Mike's only child. $Exists\ The(PP, Child(Mike, PP))$ is the proposition that the described child exists. We have

$$\begin{aligned} & true\ Exists\ The(PP, Child(Mike, PP)) \\ & \equiv true\ Exist(PP, Child(Mike, PP) \\ & \qquad And\ All(PP1, Child(Mike, PP1)\ Implies\ Equal(PP, PP1)))), \end{aligned} \tag{81}$$

but we may want the equality of the two propositions, i.e.

$$Exists\ The(PP, Child(Mike, PP))$$

$$= Exist(PP, Child(Mike, PP)$$

$$And\ All(PP1, Child(Mike, PP1)\ Implies\ Equal(PP, PP1))). \quad (82)$$

This is part of general problem of when two logically equivalent concepts are to be regarded as the same.

In order to discuss the truth of propositions and the denotation of descriptions, we introduce *possible worlds* reluctantly and with an important difference from the usual treatment. We need them to give values to the inner variables, and we can also use them for axiomatizing the modal operators, knowledge, belief and tense. However, for axiomatizing quantification, we also need a function α such that

$$\pi' = \alpha(V, x, \pi) \quad (83)$$

is the possible world that is the same as the world π except that the inner variable V has the value x instead of the value it has in π. In this respect our possible worlds resemble the *state vectors* or *environments* of computer science more than the possible worlds of the Kripke treatment of modal logic. This Cartesian product structure on the space of possible worlds can also be used to treat counterfactual conditional sentences. [6]

Let π_0 be the actual world. Let $true(P, \pi)$ mean that the proposition P is true in the possible world π. Then

$$(\forall P)(true\ P \equiv true(P, \pi_0)). \quad (84)$$

Let $denotes(X, x, \pi)$ mean that X denotes x in π, and let $denot(X, \pi)$ mean the denotation of X in π when that is defined.

The truth condition for $All(V, P)$ is then given by

$$(\forall \pi V P)(true(All(V, P), \pi) \equiv (\forall x) true(P, \alpha(V, x, \pi)). \quad (85)$$

Here V ranges over inner variables, P ranges over propositions, and x ranges over things. There seems to be no harm in making the domain of x depend on π. Similarly

$$(\forall \pi V P)(true(Exist(V, P), \pi) \equiv (\exists x) true(P, \alpha(V, x, \pi)). \quad (86)$$

[6] 1995: (McCarthy 1979) treats "Cartesian counterfactuals".

The meaning of $The(V,P)$ is given by

$$(\forall \pi V P x)(true(P, \alpha(V,x,\pi)) \wedge (\forall y)(true(P, \alpha(V,y,\pi)) \supset y = x) \\ \qquad \supset denotes(The(V,P), x, \pi)) \tag{87}$$

and

$$(\forall \pi\ V\ P)(\neg(\exists x)(true(P, \alpha(V,x,\pi)) \supset \neg true\, Exists\, The(V,P))). \tag{88}$$

We also have the following *syntactic* rules governing propositions involving quantification:

$$(\forall \pi\ Q1\ Q2\ V)(absent(V,Q1) \wedge true(All(V, Q1 Implies Q2), \pi) \\ \qquad \supset true(Q1 Implies All(V,Q2), \pi)) \tag{89}$$

and

$$(\forall \pi\ V\ Q\ X)(true(All(V,Q), \pi) \supset true(Subst(X,V,Q), \pi)) \tag{90}$$

where $absent(V,X)$ means that the variable V is not present in the concept X, and $Subst(X,V,Y)$ is the concept that results from substituting the concept X for the variable V in the concept Y. $absent$ and $Subst$ are characterized by the following axioms:

$$(\forall V1\ V2)(absent(V1,V2) \equiv V1 \neq V2), \tag{91}$$

$$(\forall\ V\ P\ X)(absent(V, Know(P,X)) \equiv absent(V,P) \wedge absent(V,X)), \tag{92}$$

axioms similar to (92) for other conceptual functions,

$$(\forall V\ Q) absent(V, All(V,Q)), \tag{93}$$

$$(\forall V\ Q) absent(V, Exist(V,Q)), \tag{94}$$

$$(\forall V\ Q) absent(V, The(V,Q)), \tag{95}$$

$$(\forall V\ X)(Subst(V,V,X) = X), \tag{96}$$

$$(\forall X\ V)(Subst(X,V,V) = X) \tag{97}$$

$$(\forall X\ V\ P\ Y)(Subst(X,V,Know(P,Y)) \\ \qquad = Know(Subst(X,V,P), Subst(X,V,Y))), \tag{98}$$

axioms similar to (98) for other functions,

$$(\forall X\ V\ Q)(absent(V,Y) \supset Subst(X,V,Y) = Y), \tag{99}$$

$$\begin{aligned}(\forall X\ V1\ V2\ Q)(&V1 \neq V2 \wedge absent(V2,X) \\ &\supset Subst(X,V1,All(V2,Q)) = All(V2,Subst(X,V1,Q)))\end{aligned} \tag{100}$$

and corresponding axioms to (100) for $Exist$ and The.

Along with these comes an axiom corresponding to α-conversion,

$$(\forall V1\ V2\ Q)(All(V1,Q) = All(V2,Subst(V2,V1,Q))). \tag{101}$$

The functions *absent* and *Subst* play a "syntactic" role in describing the rules of reasoning and don't appear in the concepts themselves. It seems likely that this is harmless until we want to form concepts of the laws of reasoning.

We used the Greek letter π for possible worlds, because we did not want to consider a possible world as a thing and introduce concepts of possible worlds. Reasoning about reasoning may require such concepts or else a formulation that doesn't use possible worlds.

Martin Davis (in conversation) pointed out the advantages of an alternate treatment avoiding possible worlds in case there is a single domain of individuals each of which has a standard concept. Then we can write

$$(\forall V\ Q)(true\ All(V,Q) \equiv (\forall x)(true\ Subst(Concept1x,V,Q)). \tag{102}$$

8 Possible Applications to Artificial Intelligence

The foregoing discussion of concepts has been mainly concerned with how to translate into a suitable formal language certain sentences of ordinary language. The success of the formalization is measured by the extent to which the logical consequences of these sentences in the formal system agree with our intuitions of what these consequences should be. Another goal of the formalization is to develop an idea of what concepts really are, but the possible formalizations have not been explored enough to draw even tentative conclusions about that.

For artificial intelligence, the study of concepts has yet a different motivation. Our success in making computer programs with *general* *intelligence* has been extremely limited, and one source of the limitation is our inability to formalize what the world is like in general. We can try to separate the problem of describing the general aspects of the world from the problem of using such a description and the facts of a situation to discover a strategy for achieving a goal. This is called separating the *epistemological* and the *heuristic* parts of the artificial intelligence problem and is discussed in (McCarthy and Hayes 1969).

We see the following potential uses for facts about knowledge:

1. A computer program that wants to telephone someone must reason about who knows the number. More generally, it must reason about what actions will obtain needed knowledge. Knowledge in books and computer files must be treated in a parallel way to knowledge held by persons.

2. A program must often determine that it does not know something or that someone else doesn't. This has been neglected in the usual formalizations of knowledge, and methods of proving possibility have been neglected in modal logic. Christopher Goad (to be published) has shown how to prove ignorance by proving the existence of possible worlds in which the sentence to be proved unknown is false. Presumably proving one's own ignorance is a stimulus to looking outside for the information. In competitive situations, it may be important to show that a certain course of action will leave competitors ignorant.

3. Prediction of the behavior of others depends on determining what they believe and what they want.

It seems to me that AI applications will especially benefit from first order formalisms of the kind described above. First, many of the present problem solvers are based on first order logic. Morgan (1976) in discussing theorem proving in modal logic also translates modal logic into first order logic. Second, our formalisms leaves the syntax and semantics of statements not involving concepts entirely unchanged, so that if knowledge or wanting is only a small part of a problem, its presence doesn't affect the formalization of the other parts.

9 Abstract Languages

The way we have treated concepts in this paper, especially when we put variables in them, suggests trying to identify them with terms in some language. It seems to me that this can be done provided we use a suitable notion of *abstract language*.

Ordinarily a language is identified with a set of strings of symbols taken from some alphabet. McCarthy (1963) introduces the idea of *abstract syntax*, the idea being that it doesn't matter whether sums are represented $a + b$ or $+ab$ or $ab+$ or by the integer $2^a 3^b$ or by the LISP S-expression (PLUS A B), so long as there are predicates for deciding whether an expression is a sum and functions for forming sums from summands and functions for extracting the summands from the sum. In particular, abstract syntax facilitates defining the semantics of programming languages, and proving the properties of interpreters and compilers. From that point of view, one can refrain from specifying any concrete representation of the "expressions" of the language and consider it merely a collection of abstract synthetic and analytic functions and predicates for forming, discriminating and taking apart *abstract expressions*. However, the languages considered at that time always admitted representations as strings of symbols.

If we consider concepts as a free algebra on basic concepts, then we can regard them as strings of symbols on some alphabet if we want to, assuming that we don't object to a non-denumerable alphabet or infinitely long expressions if we want standard concepts for all the real numbers. However, if we want to regard $Equal(X, Y)$ and $Equal(Y, X)$ as the same concept, and hence as the same "expression" in our language, and we want to regard expressions related by renaming bound variables as denoting the same concept, then the algebra is no longer free, and regarding concepts as strings of symbols becomes awkward even if possible.

It seems better to accept the notion of *abstract language* defined by the collection of functions and predicates that form, discriminate, and extract the parts of its "expressions". In that case it would seem that concepts can be identified with expressions in an abstract language.

10 Remarks and Acknowledgements

The treatment given here should be compared with that in (Church 1951b) and in (Morgan 1976). Church introduces what might be called a two-dimensional type structure. One dimension permits higher order functions and predicates as in the usual higher order logics. The second dimension permits concepts of concepts, etc. No examples or applications are given. It seems to me that concepts of concepts will be eventually required, but this can still be done within first order logic.

Morgan's motivation is to use first order logic theorem proving programs to treat modal logic. He gives two approaches. The syntactic approach—which he applies only to systems without quantifiers—uses operations like our *And* to form compound propositions from elementary ones. Provability is then axiomatized in the outer logic. His semantic approach uses axiomatizations of the Kripke accessibility relation between possible worlds. It seems to me that our treatment can be used to combine both of Morgan's methods, and has two further advantages. First, concepts and individuals can be separately quantified. Second, functions from things to concepts of them permit relations between concepts of things that could not otherwise be expressed.

Although the formalism leads in almost the opposite direction, the present paper is much in the spirit of (Carnap 1956). We appeal to his ontological tolerance in introducing concepts as objects, and his section on intensions for robots expresses just the attitude required for artificial intelligence applications.

We have not yet investigated the matter, but plausible axioms for necessity or knowledge expressed in terms of concepts may lead to the paradoxes discussed in (Kaplan and Montague 1960) and (Montague 1963). Our intuition is that the paradoxes can be avoided by restricting the axioms concerning knowledge of facts about knowledge and necessity of statements about necessity. The restrictions will be somewhat unintuitive as are the restrictions necessary to avoid the paradoxes of naive set theory.

Chee K. Yap (1977) proposes *virtual semantics* for intensional logics as a generalization of Carnap's individual concepts. Apart from the fact that Yap does not stay within conventional first order logic, we don't yet know the relation between his work and that described here.

I am indebted to Lewis Creary, Patrick Hayes, Donald Michie, Barbara Partee and Peter Suzman for discussion of a draft of this paper. Creary in particular has shown the inadequacy of the formalism for expressing all readings of the ambiguous sentence *"Pat knows that Mike knows what Joan last asserted"*. There has not been time to modify the formalism to fix this inadequacy, but it seems likely that concepts of concepts are required for an adequate treatment.

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