Cooperative Multiagent Search for Portfolio Selection

David C. Parkes
Computer and Information Science Department
University of Pennsylvania
Philadelphia, PA 19104
dparkes@unagi.cis.upenn.edu

Bernardo A. Huberman
Internet Ecologies Group
Xerox Palo Alto Research Center
Palo Alto, CA 94304
huberman@parc.xerox.com

Abstract

We present a new multiagent model for the multi-period portfolio selection problem. Individual agents receive a share of initial wealth, and follow an investment strategy that adjusts their portfolio as they observe movements of the market over time. The agents share their wealth at the end of the final investment period. We show that a multiagent system can outperform a single agent that invests all the wealth in a simple stochastic market environment. Furthermore, a cooperative multiagent system, with a simple communication mechanism of explicit hint exchange, achieves a further increase in performance. Finally we show that communication is redundant in a more realistic market that satisfies the constraints between volatility and return implied by the Capital Asset Pricing Model.

1 Introduction

Portfolios are an effective way of increasing returns while decreasing risk when investing in the stock market (28). For this reason there has been considerable attention to portfolio selection strategies in the financial (8; 12) and statistics literature (32; 10; 2; 11). As a general model for the decision faced by a computational agent with limited resources that acts in an uncertain environment, portfolio-selection techniques have recently been applied to new problem domains: the selection of portfolios of heuristics for solving hard computational problems (17); and portfolio strategies for message passing to reduce risk in uncertain communication domains (27).

We introduce a new multiagent model for portfolio selection that builds on a recent computationally-efficient portfolio selection strategy with a worst-case performance guarantee (15). The multiagent model assumes a system of bounded-rational cooperative computational agents that pool their initial wealth, manage a share of the investment each, and then pool their final wealth. The agents use a myopic strategy to change their portfolio between investment periods, based on the current market prices and their current portfolio. We later allow the agents to communicate through the exchange of the recent performance of their portfolio selection strategies. An agent can switch to the portfolio strategy of the agent that has been performing best in the recent past. This simple mechanism of "hint exchange" has enabled exponential performance improvements in other cooperative problem solving domains (18; 9).

We derive a new interpretation of the multi-period portfolio selection problem as search through portfolio space, where an agent explores a new state in each investment period. We present the results of a quantitative assessment of the performance of our multiagent portfolio selection model in a simple stochastic market that show that: (a) a system of independent agents will outperform a single agent; (b) a system of agents can further improve their performance by sharing short-term

portfolio strategies. This confirms that cooperative multiagent search improves portfolio selection through efficient search. Finally, we show that communication through hint exchange is redundant in stochastic markets that satisfy the Capital Asset Pricing Model (CAPM). This model places constraints on the volatility of stock dynamics, imposing correlations between the price movements of individual stocks. The CAPM model is a more realistic market model, and this result suggests that communication is the mechanism that leads to the observed dynamics and efficiencies in real markets.

2 Multi-Period Portfolio Selection

In this section we introduce a formal model of the portfolio selection problem in a stochastic stock market. Given this model, the traditional economic approach to portfolio selection selects optimal portfolios over time through direct optimization, while modern portfolio theory suggests a single-period mean-variance approximation. In both of these approaches strong assumptions are made about the underlying statistics of the market. The portfolio selection strategy that is implemented by the individual agents in our multiagent model is *model-free*, and its performance is robust to specific assumptions about the statistics of a market.

A portfolio in a market of N stocks in a single investment period is represented as a vector $\mathbf{w} = (w_1, \ldots, w_N)$, where $w_i \geq 0$ and $\sum_{i=1}^{N} w_i = 1$. A fraction w_i of wealth is invested in stock i at the start of the period. The total change in wealth over the period depends on the change in price of the stocks held in the portfolio. Given a vector of "price relatives", $\mathbf{x} = (x_1, \ldots, x_N)$, where x_i is ratio of closing price to opening price over the period for stock i, then the wealth of an agent with portfolio $\mathbf{w}$ increases (or decreases) by a factor of $\mathbf{w} \cdot \mathbf{x} = \sum_{i=1}^{N} w_i x_i$. This is the simple gross return from portfolio $\mathbf{w}$. The standard *multi-period portfolio selection problem* is chooses a sequence of portfolios $\{\mathbf{w}^T\} = (\mathbf{w}^1, \ldots, \mathbf{w}^T)$ to maximize the expected utility over return on investment, given a sequence of price relatives sampled from a stationary distribution, $\{\mathbf{x}^T\} = (\mathbf{x}^1, \ldots, \mathbf{x}^T)$. The return on investment, $\mathcal{R}_{\mathcal{S}}$ from portfolio selection strategy $\mathcal{S}$, after T periods is $\mathcal{R}_{\mathcal{S}} = \prod_{t=1}^{T} \mathbf{w}_{\mathcal{S}}^t \cdot \mathbf{x}^t$, where $\mathbf{w}_{\mathcal{S}}^t$ denotes the investment portfolio in period t. A portfolio selection strategy maps a history of stock price observations to a portfolio selection for the next investment period. Given a utility function $U(R)$ over end-period return on investment, the traditional economic approach to multi-period portfolio selection is to follow a strategy that generates a sequence of portfolios $\{\mathbf{w}^T\}$ to solve

$$\max_{\{\mathbf{w}^T\}} E_{\{\mathbf{X}^T\}} \left[U \left(\prod_{t=1}^{T} \mathbf{w}^t \cdot \mathbf{x}^t \right) \right] \tag{1}$$

The optimal portfolio strategy will depend on the risk-preferences of the investor. Typically investors are risk-averse, with concave increasing utility functions over final wealth (7). A good investment strategy makes a tradeoff between expected final-period wealth and variance in final-period wealth to maximize expected utility. Non-linear programming techniques can be used to solve this optimization problem for a restricted class of utility functions, given a statistical model for the future dynamics of the stock market (3).

Modern portfolio theory introduces approximate "mean-variance" analysis to simplify the portfolio selection problem (28). The "risk" of a portfolio is quantified as the standard deviation of return from period to period, and the portfolio selection problem is reduced to computing an "efficient" portfolio that minimizes risk for a fixed level of return, in a single period. While this approach is mathematically and computationally tractable, it still requires that an investor first *estimates* model parameters that characterize the dynamics of the stock market, and then computes the optimal portfolio selection strategy given the model. The accuracy of the underlying stock-market model and statistics are critical. For example, while a portfolio may be efficient with respect to a particular set of beliefs about the future dynamics of stock prices, its *ex post* efficiency is highly dependent on the accuracy of those beliefs. The parameter estimation problem for an economic random-variable is difficult in general (7; 8).

2.1 Model-Free Portfolio Selection Strategies

A recent game-theoretic approach to portfolio-selection designs "universal" strategies that make no statistical assumptions about the underlying stock prices, side-step specific modeling assumptions, and avoid parameter estimation problems (12). One such strategy, Exponentiated Gradient (EG) (15), presents a period-to-period update rule for an agent to adjust its portfolio, without forming an explicit model of the market. An agent updates its portfolio on the basis of its recent performance, and the stock price changes in the previous period. The χ^2 strategy (15), a first-order approximation to EG, generates the portfolio for the next period, $\mathbf{w}^{t+1}$, given the current portfolio $\mathbf{w}^t$ and recent price relatives $\mathbf{x}^t$, according to the simple update-rule

$$w_i^{t+1} = w_i^t \left(\eta(\frac{x_i^t}{\mathbf{w}^t \cdot \mathbf{x}^t} \Leftrightarrow 1) + 1 \right) \tag{2}$$

where η, which we take as positive, is the "learning rate". The update-rule increases the proportion of wealth invested in stocks that outperformed the portfolio in the previous period, and decreases the proportion of wealth invested in stocks that under-performed the portfolio, i.e. $w_i^{t+1} > w_i^t \Leftrightarrow x_i^t > \mathbf{w}^t \cdot \mathbf{x}^t$. A small learning rate will cause $\mathbf{w}^{t+1}$ to move slowly towards an optimal portfolio strategy in a stationary market with little sensitivity to period-to-period fluctuations, and a large learning rate will cause $\mathbf{w}^{t+1}$ to move more quickly tow

long-term optimal CRP is ill-defined, and (assuming periodic quasi-stationarity) it is again the speed of convergence, to the short-term optimal CRP, that is important.

We conjectured that through (a) parallel agent search; and (b) promoting cooperative search through hint exchange, the agents in our multiagent investment model would converge to the optimal portfolio more quickly than either a single agent, or a system of independent agents. This collective search mechanism has been an extremely successful strategy in other hard problem solving domains (9). Viewed as search, the performance of a portfolio selection strategy depends on the utility of the sequence of states explored during the first T iterations. The particular market model determines the utility-structure of the search space, and the distribution of input problems. Some market models can be expected to present more difficult search problems than other market models.

We present quantitative results for a medium-term multi-period investment problem in a simulated market with stationary statistics. We measure the performance $Perf_{\mathcal{S}}$, of a portfolio selection strategy $\mathcal{S}$ after T investment periods as the end-period log return on investment, averaged over J trials:

$$Perf_{\mathcal{S}} = \frac{1}{J} \sum \log \left(\mathrm{R}_{\mathcal{S}}(j) \right) \tag{3}$$

where $R_{\mathcal{S}}(j)$ is the return from strategy $\mathcal{S}$ in the j^{th} trial. We also compute the optimal CRP for the simulated markets, that is the constant rebalanced portfolio that maximizes performance for a large number of trials given knowledge of the statistics of the market.

3 Cooperative Multiagent Search for Portfolio Selection

In this section we present the results of a quantitative analysis that compares, for a simple stock market model, the performance of a system of non-adaptive, adaptive, and adaptive and communicating agents as the number of agents in the model increases. The non-adaptive agents maintain the same (random) constant-portfolio across all investment periods, trading to rebalance the portfolio from period to period. The adaptive agents receive a random initial portfolio and invest from period to period according to the χ^2 portfolio selection strategy. The communicating adaptive agents also exchange portfolio strategies and can switch to the portfolio strategy of another agent. The agents post their current strategy and its recent performance to a central "blackboard", which is read by all agents. Recent performance is measured as the return on investment achieved with the portfolio selection strategy over the past τ investment periods, termed the "performance-window". An agent will choose to switch to the portfolio strategy of the agent with the best recent performance with fixed probability p, termed the "switching probability". The cooperation parameters for each agent are drawn from a distribution that is optimized off-line for the market volatility, the size of the multiagent system, and the number of investment periods (see Section 3.2).

An agent that switches to the current portfolio of another agent in the system will approximate the *strategy* of that agent because the agents all use the same history-free χ^2 update-rule (with different learning rates) to adjust the portfolio on the basis of current prices. We do not limit the number of times that an agent can change strategies, other than forcing an agent to use a new strategy for at least τ investment periods before posting to the blackboard or switching to another strategy. We conjectured that this exchange of recently successful strategies and random switching between strategies would cause the overall portfolio selection strategy of the multiagent system to move more quickly (on average) than a single agent towards an optimal strategy. The model shows how a group of investors might behave in a complex and uncertain environment[2].

3.1 Quantitative Results

We initially simulated a market of N geometric Brownian motion stocks with normally distributed price relatives, $\mathbf{x} = (x_1, \ldots, x_N)$. The first and second moments of the distribution for each stock, $\mathbf{X}_i \sim N(\mu_i, \sigma_i^2)$, are represented by the vectors $\mu = (\mu_1, \ldots, \mu_N)$, and $\sigma = (\sigma_1, \ldots, \sigma_N)$ respectively.

[2]Of course, there is nothing to prevent one agent modeling a cooperative multiagent system internally for a small and completely observable market space.

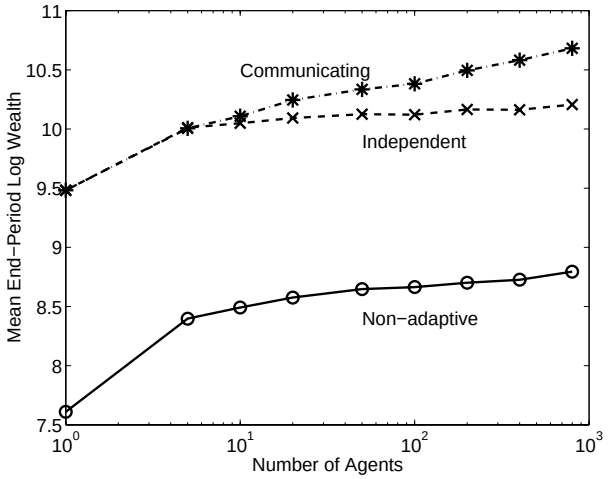


Figure 1: The performance of the non-adaptive, adaptive non-communicating, and adaptive and communicating agents as a function of the number of agents in the mutliagent portfolio selection system.

Geometric Brownian motion is often used to model the dynamics of stock prices (13). It satisfies the "Efficient Market Hypothesis" (EMH), which holds that an informationally efficient market has random price changes, and denies the possibility of "beating the market" (8). The probability distribution in geometric Brownian motion over all future prices depends only on the current price relatives, and therefore the history of past price changes carry no future predictive value.

We simulate a market of 10 stocks, over 2000 investment periods. In order to assure the statistical significance of our results we averaged the performance of each multiagent portfolio selection model over 2000 independent market trials. The stochastic parameters for each trial are drawn from uniform distributions, $\mu_i \sim U(0.9995, 1.01)$, and $\sigma_i \sim U(0.0, 0.2)$. These statistics are appropriate for the monthly returns on real stocks. For example, the mean monthly return on stock in IBM between 1962 and 1994 was 1.0081, and the standard deviation in monthly return was 0.062 (8, Page 21). In each trial we first generate the stochastic parameters, and then the stock prices. The investment models are all compared on the same sequences of stock prices.

The number of agents in our model varies between 1 and 800, with the same initial wealth shared equally among all agents for all models and trials. We assign a random initial portfolio to each agent, and allow each adaptive agent to use a different learning rate, $\eta \sim U(0.1, 0.15)$. This distribution of learning rates was found experimentally to give good performance for a wide range of multiagent model sizes, and helps to maintain a diversity of strategies within the system. In general the choice of learning-rate represents a classic tradeoff between return and risk. A high learning rate enables adaptive agents to perform well on average, but with a high chance of performing worse than non-adaptive agents (see Section 3.2). The switching rate and performance window size are the same for every agent within a system, and optimized for the number of agents, with switching-probability $p = 0.004$ and performance-window $\tau = 400$ typical.

The performance of each model is compared in Figure 1. We see that: (a) a single adaptive agent outperforms a single non-adaptive agent; (b) a system of independent adaptive agents outperforms a single adaptive agent; (c) a system of adaptive communicating agents outperforms a system of adaptive non-communicating agents for large numbers of agents. We also compute the performance of an agent that invests in the long-term optimal constant rebalanced portfolio across all trials. This optimal strategy (which requires knowledge of the statistics of the market) yields an average end-period log wealth, $Perf_{\mathbf{w}^*} = 16.0$.

The value of communication within our multiagent model for portfolio selection increases with the number of agents in the system. The difference in performance between the cooperative and

independent models is significant for systems with more than 50 agents[3]. This confirms that cooperative parallel agent search for the optimal portfolio selection strategy is more efficient than single agent search or independent parallel search in a simple stochastic market.

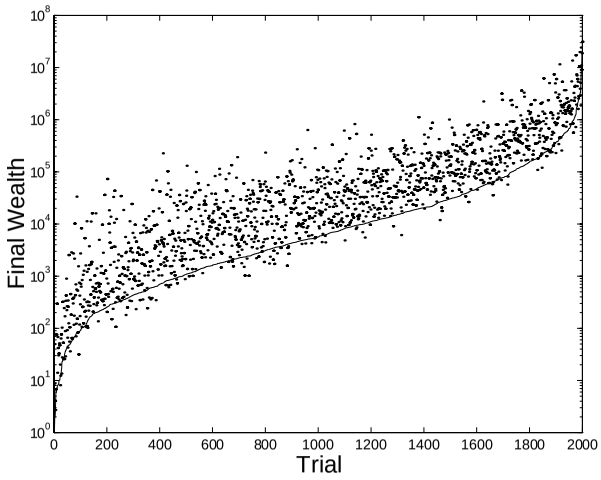


Figure 2: Final system wealth (log-scale) of 100 adaptive agents (dots) and 100 non-adaptive agents (line), over 2000 trials. The trials are sorted by final wealth of the non-adaptive agents.

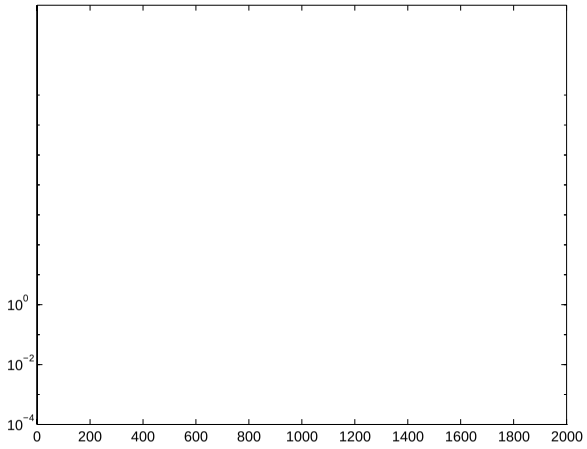
Figure 2 compares the final system wealth of 100 adaptive agents (dots) and 100 non-adaptive agents (line) in 2000 market trials. The trials are sorted by the final wealth of the non-adaptive system of agents. The adaptive agents clearly outperform the non-adaptive agents: achieving more wealth in 93% of the trials, and a final wealth that is 4.3 times greater on average. Figure 6(a) illustrates the *additional* value of communication for a system with 400 adaptive agents. The communicating agents outperform the non-communicating agents, achieving a greater wealth in 75% of the trials, with a wealth that is 1.47 times greater on average.

The slow improvement in performance for the system of non-adaptive agents as the number of agents increases shows the effect of simple diversification. Each agent invests in a new random constant portfolio strategy. Theoretically, as the number of non-adaptive agents gets very large the performance will approximate the worst-case optimal performance of a *single* adaptive agent, but the number of agents required is very large – estimated to be at least 10^9 for 10 stocks (15).

3.2 The Choice of Model Parameters

The choice of learning rate, η, for the adaptive agents represents a classic tradeoff between return and risk. When choosing a learning rate the appropriate measure of risk is the chance that an agent might perform *worse* than with no adaptive behavior at all. As the learning rate is increased, performance increases but so does risk. As a graphical illustration of the effect of learning rate on risk and performance, consider Figure 3, which compares the final system wealth of 100 adaptive agents (dots) with 100 non-adaptive agents (line), for a high learning rate, $\eta \in [0.9, 0.95]$. Figure 2 plots the same results for agents with a smaller learning rate, $\eta \in [0.1, 0.15]$. Visually the "dots" (adaptive) beat the "line" (non-adaptive) by a larger margin in Figure 3, but also fall below the line more frequently, and by more. The adaptive multiagent system with high learning rates achieves a wealth that is on average 26 times greater than that of a system of non-adaptive agents, but performs more than 80% worse in 16.5% of the trials (high performance, high risk). In comparison, the multiagent system with low learning rates outperforms a non-adaptive multiagent system by an

[3]The null hypothesis that the mean end period log wealth for a system of communicating agents and a system of independent agents is equal is rejected with a significance level of less that 0.01 for systems with more than 50 agents.


10^{0}
10^{-2}
10^{-4}
0
200
400
600
800
1000
1200
1400
1600
1800
2000

market model, is around $p = 0.008$ and $\tau = 500$. We can see that the performance drops off when the window size is too large, in this case there is little strategy switching and the system performance is similar to a system of independent agents. Similarly, when the window size is too small and the switching probability is too large, we have too much switching, too early, and we lose the advantages that come from having agents with diverse strategies. The multiagent system simulates the performance of a single agent. We optimized the cooperation parameters off-line for each number of agents, to maximize any benefit from cooperation. The optimal parameters for other multiagent system sizes are similar, with a trend to larger switching probabilities as the number of agents increases.

4 Portfolio Selection in CAPM Markets

In this section we present the results of a quantitative analysis that compares the performance of adaptive multiagent systems with and without communication in a more realistic market model. We simulate the Capital Asset Pricing Model (CAPM), which models an equilibrium for mean-variance investors with homogeneous beliefs (35; 8). Although there is some empirical evidence against CAPM the CAPM does explain a significant fraction of the price dynamics observed in real stock markets (4; 29). The dynamics of stocks in real markets are in fact highly, but not perfectly, correlated. It is this partial correlation that allows diversification through portfolio investments to reduce, but not eliminate, risk (7; 35).

The CAPM augments the simple geometric Brownian motion model with quantified correlations between stock prices. The key result of CAPM is that the expected excess return of a stock is proportional to the covariance of its return with the "market portfolio". The market portfolio is simply the result of a "buy and hold" policy that invests equally in all stocks (14). In equilibrium, stocks with high expected returns have high volatility, while stocks with low expected returns have low volatility.

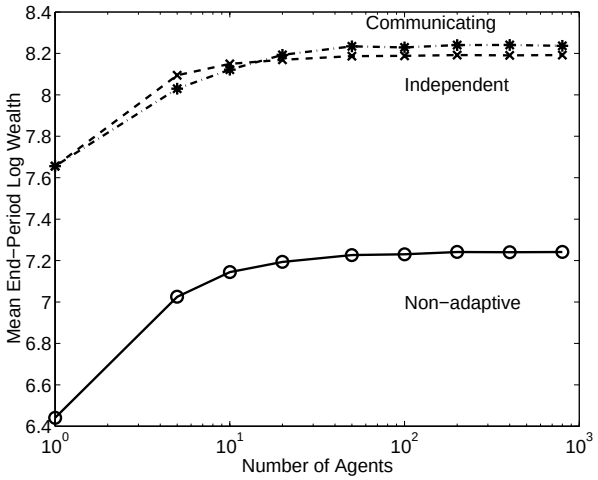
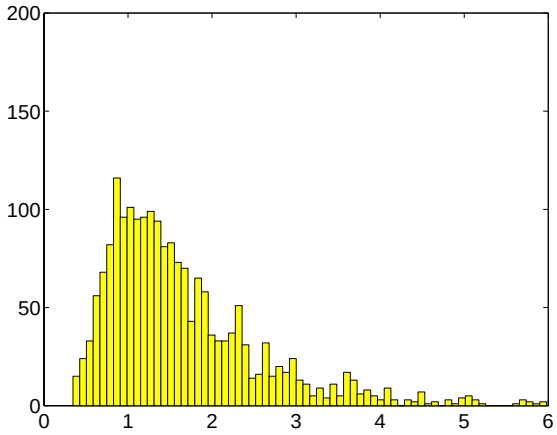


Figure 5: The performance of the non-adaptive, adaptive non-communicating, and adaptive and communicating agents as a function of the number of agents, in a CAPM market.

To simulate a CAPM for N stocks and T investment periods we generate means and variances for the marginal Normal distributions for the price relatives of each stock from the same distribution as in the simple market model[4]. We then assign covariances to satisfy constraints between the volatility and return of stocks, and complete a multivariate Normal distribution that generates a sequence of stock prices with statistics that fall approximately onto the "security market line" (35), such that the excess return of each stock is proportional to its covariance with the market portfolio.

[4]We choose not to include a risk-free asset in our model of CAPM. We justify this by assuming that the simulated returns on stocks are already "excess-returns" over the risk-free return.


200
150
100
50
0
0
1
2
3
4
5
6

between investor actions and price dynamics to predict equilibrium statistics, that makes further communication worthless.

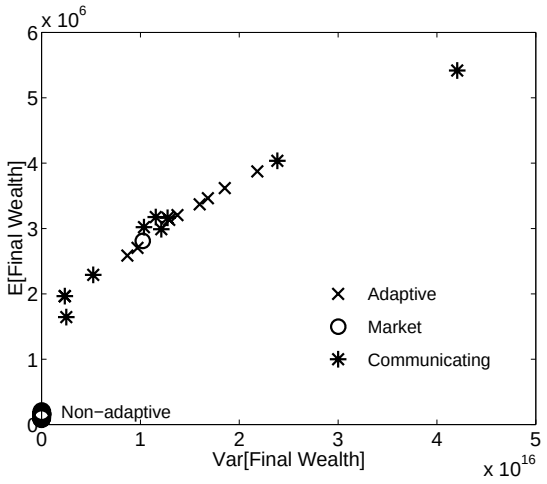


Figure 7: Expected return versus variance in return for the Market portfolio, and various sizes of systems of non-adaptive agents, adaptive agents, and adaptive and communicating agents. All adaptive portfolio selection-strategies, and the market portfolio, lie on the same line in mean-variance space.

Finally, we compare the performance of our multiagent model to the performance of the market portfolio. In a market that satisfies CAPM, such as the second set of simulated markets, all adequately diversified portfolios, including the market portfolio, will have the same "Sharpe ratio", ratio of excess expected return to variance in return (35). Figure 7 shows that this is the case, the overall portfolios of the multiagent portfolio investment systems and the market portfolio all plot along the same line in mean-variance space. Only the non-adaptive agents are less mean-variance efficient than the market, due to a lack of diversification.

However, the independent multiagent model outperforms the market in terms of expected utility for an investor with a logarithmic utility function over final wealth. Figure 8 compares the distribution of the logarithm of final wealth for the market portfolio and a system of 400 adaptive agents. We see that the multiagent system of independent agents is able to significantly outperform the market portfolio, achieving a mean log-wealth of 8.19, while the market only achieves a mean log-wealth of 5.31, despite being mean-variance efficient. Indeed, the buy and hold strategy of the market portfolio performs worse than the average performance of a single investor with a random constant rebalanced portfolio (see Figure 5).

Modern portfolio theory reduces portfolio selection to the set of portfolios that lie on the "efficient frontier" in mean-variance space, but provides no insight into how to select between efficient portfolios. All the adaptive portfolio strategies, and the market portfolio, lie on the efficient frontier

Table 1: Relative performance over 2000 trials in the standard market model and the CAPM market model, for multiagent systems with 800 agents.

Market Model	Investment Model		
	Non-adaptive agents	Adaptive agents	Communicating and adaptive agents
Simple	55%	64%	67%
CAPM	59%	67%	67%

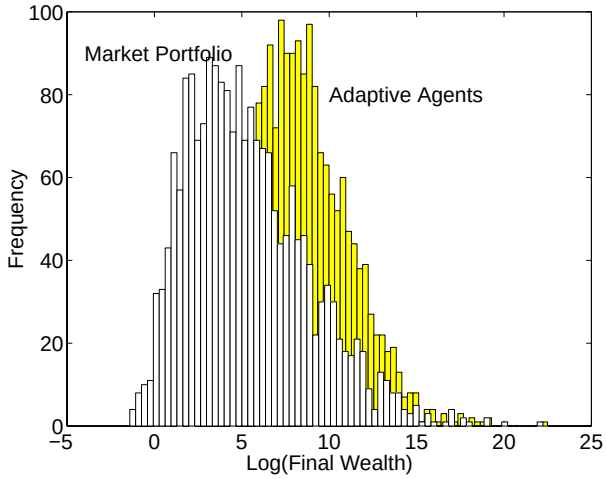


Figure 8: Distribution of final log wealth of the market portfolio and a system of 400 adaptive agents in a simulated CAPM market.

in Figure 7. Although we can expect the performance of the market portfolio to improve through borrowing (or lending) a risk-free asset to move the overall portfolio statistics on the efficient frontier, the relatively poor performance of the market portfolio is also explained by a closer inspection of the distributional properties of the final wealth from the market portfolio and the multiagent portfolio selection models.

Table 2: Correlation of final wealth with the best CRP wealth over 2000 trials in a simulated CAPM market, for multiagent systems with 800 agents.

Investment Model			
Market Portfolio	Non-adaptive agents	Adaptive agents	Communicating and adaptive agents
0.1281	0.6872	0.9988	0.9956

The ratio of the first two moments of a distribution is not a sufficient statistic with which to compare the expected log of a distribution. There are other important distributional differences, and we get some insight by comparing the correlation of final wealth with the end-period wealth of the best CRP in each trial across portfolio selection strategies. Table 2 shows that while the performance of the market portfolio remains almost uncorrelated with the best CRP across multiple trials, the adaptive agents are able to achieve a wealth that is almost perfectly correlated with the wealth of the best CRP strategy. The agents are able to "boost" the performance at the tail of the wealth distribution by tracking the best possible gain that they can achieve very closely.

5 Related Work

To the best of our knowledge this is the first work to consider the performance of a system of multiple adaptive agents for the portfolio-selection problem. Blum and Kalai (5) recognize that a system of non-adaptive agents will approximate the worst-case optimal performance of a single EG-adaptive agent as the number of agents gets large, but do not consider either an adaptive multiagent system, or the effects of cooperation.

There has been previous work on using multiple heuristics to solve search problems: sequential methods with possible restart (33; 26; 19; 6); parallel independent methods (31; 25; 20; 23; 17); and cooperative parallel multiagent search (22; 1; 16; 9). A general theory predicts superlinear speedup

in the performance of individual agents when the search methods are diverse and the agents are able to utilize information found in other parts of the search space (18).

6 Conclusions and Future Work

In this paper we have introduced a new multiagent model for portfolio selection that mixes parallel search with hint exchange. The model assumes a system of bounded rational cooperative agents that pool their initial wealth, each manage a share, and then pool their final wealth. The quantitative results show that a system of adaptive agents with simple update-rules, that start with random portfolios and exchange portfolio strategies with good recent performance, will outperform a single adaptive agent in a simple market model with no global structure relating the expected return and volatility of each stock.

These results are also applicable to economic approaches to hard computational problems, where it has been shown that a suitable portfolio of heuristics can improve the performance of programs for solving very hard problems (17). If individual processes choosing among possible heuristics are allowed to communicate, the final portfolio to which they converge will have the same optimal characteristics as the one we considered in this paper.

When the market statistics have more structure, such as in the CAPM market, an adaptive multiagent system will still outperform a system of non-adaptive agents or a single adaptive agent. However, communication between the agents becomes redundant, and cooperating agents do no better than independent agents. Finally, we showed that while the "market portfolio" that invests across all stocks equally will achieve an optimal ratio of expected wealth to variance in wealth, its performance in terms of expected end-period log wealth is worse than that of our multiagent portfolio selection model. The end-period wealth from the independent multiagent cooperative selection models is highly correlated with end-period wealth of the best CRP, and we believe that this favorably skews the distribution of end-period wealth.

In future work we will investigate how the performance of our cooperative multiagent portfolio selection model scales with the number of stocks in the market space. We also propose further analysis of the micro- and macro-properties of the search algorithm that is implemented by the multiagent portfolio selection model, focusing at the micro-level at the occurrence and frequency of strategy switching between the agents, and at the macro-level on the efficiency of the search algorithm through aggregate portfolio space.

7 Appendix

In this appendix we prove a number of optimality properties for the constant rebalanced portfolio (CRP) that optimizes asymptotic per-period return in a stationary stochastic market:

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \lim_{T\to\infty} \left(\prod_{t=1}^{T} \mathbf{w} \cdot \mathbf{x}^t \right)^{1/T} \tag{4}$$

where, $\mathbf{w} = (w_1, \ldots, w_N)$ represents a constant rebalanced portfolio across N stocks, with investment w_i maintained in stock i across all investment periods, $w_i \geq 0$, $\sum_N w_i = 1$; $\mathbf{x}^t = (x_1^t, \ldots, x_N^t)$ represents the price relatives in period t, x_i^t is the ratio of closing price to opening price of stock i in period t, i.i.d. across periods; T is the number of investment periods.

Claim 1. *The best CRP, $\mathbf{w}^*$, also maximizes expected single period log return.*

Proof.

$$\mathbf{w}^* = \arg\max_{\mathbf{w}} \lim_{T\to\infty} \left(\prod_{t=1}^{T} \mathbf{w} \cdot \mathbf{x}^t \right)^{1/T}$$

$$
\begin{aligned}
&= \arg\max_{\mathbf{w}} \lim_{T\to\infty} \left(\frac{1}{T} \sum_{t=1}^{T} \log \mathbf{w} \cdot \mathbf{x}^t \right) \\
&= \arg\max_{\mathbf{w}} E_{\mathbf{X}} \log \mathbf{w} \cdot \mathbf{x} \qquad \square
\end{aligned}
$$

Claim 2. *The CRP that maximizes expected single period log return also maximizes expected end period log return, asymptotically for large numbers of investment periods.*

Proof.

$$
\begin{aligned}
\mathbf{w}^* &= \arg\max_{\mathbf{w}} \lim_{T\to\infty} \left(E_{\{\mathbf{X}^T\}} \log \prod_{t=1}^{T} \mathbf{w} \cdot \mathbf{x}^t \right) \\
&= \arg\max_{\mathbf{w}} \lim_{T\to\infty} \left(E_{\{\mathbf{X}^T\}} \sum_{t=1}^{T} \log \mathbf{w} \cdot \mathbf{x}^t \right) \\
&= \arg\max_{\mathbf{w}} \lim_{T\to\infty} \left(\sum_{t=1}^{T} E_{\mathbf{X}} \log \mathbf{w} \cdot \mathbf{x} \right) \\
&= \arg\max_{\mathbf{w}} E_{\mathbf{X}} \log \mathbf{w} \cdot \mathbf{x} \qquad \square
\end{aligned}
$$

Claim 3. *The CRP that maximizes expected single period log return also maximizes expected end period log return for any number of investment periods.*

Proof.

$$
\begin{aligned}
\mathbf{w}^* &= \arg\max_{\mathbf{w}} E_{\{\mathbf{X}^T\}} \log \prod_{t=1}^{T} \mathbf{w} \cdot \mathbf{x}^t \\
&= \arg\max_{\mathbf{w}} E_{\{\mathbf{X}^T\}} \sum_{t=1}^{T} \log \mathbf{w} \cdot \mathbf{x}^t \\
&= \arg\max_{\mathbf{w}} \sum_{t=1}^{T} E_{\mathbf{X}} \log \mathbf{w} \cdot \mathbf{x} \\
&= \arg\max_{\mathbf{w}} E_{\mathbf{X}} \log \mathbf{w} \cdot \mathbf{x} \qquad \square
\end{aligned}
$$

Claim 4. *Any portfolio selection strategy $\mathcal{S}$ that converges to the best CRP in a finite number of investment periods will achieve an optimal per-period growth rate asymptotically, as the number of investment periods gets large.*

Proof. We prove (equivalently, from Claim 1) that the average per-period log return from strategy $\mathcal{S}$ approaches the optimal expected per-period log return as the number of investment periods, T, gets large. Let T_1 represent the number of periods that pass before strategy $\mathcal{S}$ selects the optimal CRP, $\mathbf{w}^*$; μ_1 denote the average per-period log return received during those periods; and μ^* denote the expected per-period log return from $\mathbf{w}^*$. Then the average per-period log return from strategy $\mathcal{S}$ as the number of investment periods gets large is

$$
\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \log \mathbf{w}_{\mathcal{S}} \cdot \mathbf{x}^t \\
&= \lim_{T\to\infty} \frac{1}{T} \left(T_1 \mu_1 + (T \Leftrightarrow T_1) \mu^* \right) \\
&= \mu^* \qquad \square
\end{aligned}
$$

References

[1] Aldous, D., and Vazirani, U. 1994. "Go with the winners" algorithms. In *Proc. of the 35th Symp. on Found. of Comp. Sci.*, 492–501.

[2] Algoet, P. H., and Cover, T. M. 1988. Asymptotic optimality and asymptotic equipartition properties of log-optimum investment. *The Annals of Probability* 16(2):876–898.

[3] Bertsekas, D. P. 1987. *Dynamic Programming: Deterministic and Stochastic Models.* Prentice-Hall.

[4] Black, F., Jensen M. C., and Scholes M. 1972. The Capital Asset Pricing Model: Some Empirical Tests. In Jensen M. C., Ed. *Studies in the Theory of Capital Markets.* 1972 Praeger, New York.

[5] Blum, A., and Kalai, A. 1997. Universal portfolios with and without transaction costs. In *Proceedings of the Tenth Annual Conference on Computational Learning Theory*, 309–313.

[6] Boese, K. D., Kahng, A. B., and Muddu, S. 1994. A new adaptive multi-start technique for combinatorial global optimizations. *Operations Research Letters* 16:101–113.

[7] Borch, K. H. 1968. *The Economics of Uncertainty.* Princeton University Press.

[8] Campbell, J. Y., Lo, A. W., MacKinlay, C. 1997. *The Econometrics of Financial Markets.* Princeton University Press, Princeton NJ.

[9] Clearwater, S. H., Huberman, B. A., and Hogg, T. 1991. Cooperative solution of constraint satisfaction problems. *Science* 254:1181–1183.

[10] Cover, T. M., and Gluss, D. H. 1986. Empirical Bayes stock market portfolios. *Advances in Applied Mathematics* 7:170–181.

[11] Cover, T. M., and Ordentlich, E. 1996. Universal portfolios with side information. *IEEE Transactions on Information Theory* 42(2):348–363.

[12] Cover, T. M. 1991. Universal portfolios. *Mathematical Finance* 1(1):1–29.

[13] Dixit, A. K., and Pindyck, R. S. 1994. *Investment under Uncertainty.* Princeton University Press, Princeton NJ.

[14] Garbade, K. 1982. *Securities Markets.* McGraw-Hill.

[15] Helmbold, D. P., Schapire, R. E., Singer, Y., and Warmuth, M. K. 1996. On-line portfolio selection using multiplicative updates. In *Machine Learning: Proceedings of the Thirteenth International Conference.* Revised July, 1997.

[16] Hogg, T., and Williams, C. P. 1993. Solving the really hard problems with cooperative search. In *Proc. 11th National Conference on Artificial Intelligence (AAAI-93)*, 231–236.

[17] Huberman, B. A., Lukose, R. M., and Hogg, T. 1997. An economics approach to hard computational problems. *Science* 275:51–54.

[18] Huberman, B. A. 1990. The performance of cooperative processes. *Physica D* 42:38–47.

[19] Johnson, D. S., Aragon, C. R., McGeoch, L. A., and Schevon, C. 1989. Optimization by simulated annealing: an experimental evaluation. Part I, graph partitioning. *Operations Research* 37:865–892.

[20] Kauffman, S., and Levin, S. 1987. Toward a general theory of adaptive walks on rugged landscapes. *Journal of Theoretical Biology* 128:11–45.

[21] Kivinen, J., and Warmuth, M. K. 1994. Exponentiated gradient versus gradient descent for linear predictors. Technical Report UCSC-CRL-94-16, University of California, Santa Cruz. Revised December,1995.

[22] Knight, K. 1993. Are many reactive agents better than a few deliberative ones? In *Proc. 13th International Joint Conference on Artificial Intelligence (IJCAI-93)*, 432–437.

[23] Kornfeld, W. A. 1981. The use of parallelism to implement a heuristic search. In *Proc. 7th International Joint Conference on Artificial Intelligence (IJCAI-81)*, 575–580.

[24] Lintner, J. 1965. The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *Review of Economics and Statistics* 47:13–37.

[25] Luby, M., and Ertel, W. 1993. Optimal parallelization of Las Vegas algorithms. Technical Report TR-93-041, International Computer Science Institute, Berkeley, CA.

[26] Luby, M., Sinclair, A., and Zuckerman, D. 1993. Optimal speedup of Las Vegas algorithms. Technical Report TR-93-010, International Computer Science Institute, Berkeley, CA.

[27] Lukose, R., and Huberman, B. 1997. A methodology for managing risk in electronic transactions over the Internet. In *3rd Int. Conf. on Computational Economics.*

[28] Markowitz, H. M. 1959. *Portfolio Selection.* Wiley, New York.

[29] Merton, R. C. 1997. *Continuous-Time Finance.* Blackwell, MA.

[30] Ordentlich, E., and Cover, T.M. 1996. The cost of achieving the best portfolio in hindsight. Technical Report NSF-90, Stanford University.

[31] Rao, V., and Kumar, V. 1992. On the efficiency of parallel backtracking. *IEEE Trans. on Parallel and Dist. Systems.*

[32] Samuelson, P. A. 1969. Lifetime portfolio selection by dynamic stochastic programming. *Review of Economics and Statistics* 51:239–246.

[33] Selman, B., Levesque, H., and Mitchell, D. 1992. A new method for solving hard satisfiability problems. In *Proc. 10th National Conference on Artificial Intelligence (AAAI-92)*, 440–446.

[34] Sharpe, W. F. 1964. Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk. *Journal of Finance.* 19:425–442.

[35] Sharpe, W. F. 1970. *Portfolio Theory and Capital Markets.* McGraw-Hill.